J Eng Math (2008) 61:171–184
DOI 10.1007/s10665-007-9154-6

Analysis of 2D non-axisymmetric elasticity and thermoelasticity problems for radially inhomogeneous hollow cylinders

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Received: 16 October 2006 / Accepted: 10 May 2007 / Published online: 27 June 2007

Abstract An analytical approach to solve plane static non-axisymmetric elasticity and thermoelasticity problems for radially inhomogeneous hollow cylinders is presented. This approach is based upon the direct integration method proposed by Vihak (Vigak). The essence of the method mentioned is in the integration of the original differential equilibrium equations, which are independent of the stress–strain relations. This gives the opportunity to deduce the relations, which are invariant with respect to various properties of the material, for the stress-tensor components. From these relations each of the stress-tensor components have been expressed in terms of the governing one. A solution of the equation for the governing stress in the form of Fourier series is presented. To determine the Fourier coefficients, an integral Volterra-type equation is derived and solved by a simple iteration method with rapid convergence. Other stress-tensor components are expressed through the obtained governing stress in the form of an explicit functional dependence on force and thermal loadings.

Keywords Plane elasticity · Radially inhomogeneous hollow cylinder · Stresses · Thermoelasticity

1 Introduction

One of most important actual problems in the linear theories of elasticity and thermoelasticity is the solution of the boundary-value problem for inhomogeneous (mechanical properties depend on the space coordinates) solids. Such materials are widely used in modern technology. As an example, we mention functionally graded materials, whose properties can be formed technologically during the manufacturing process with the goal of optimizing their characteristics; composites, etc. It is well known that, if one does not take such kinds of dependence of the material properties of real objects into account, this can cause huge errors in the computed stresses. At the same time, the solution of the elasticity and thermoelasticity problem in a rather general formulation, under the prescribed heating regime and force loading, is complicated.

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The question of solving elasticity problems for a homogeneous hollow cylinder is investigated in many popular monographs on the theories of elasticity and thermoelasticity. Some particular cases of force loadings given on cylinder surfaces (as in the case of normal and tangential tractions uniformly distributed on inner and outer surfaces; compression of a cylinder by two opposite concentrated forces or by uniform forces distributed on some parts of cylinder's boundary, etc.) are considered and widely discussed by Timoshenko and Goodier [1], Little [2], Lurie [3], Papkovich [4], Sadd [5], and many others. Moreover, the references [1–5], and others give the general solution in Michell's form of the biharmonic equation for such problems. This solution contains enough degrees of freedom to satisfy boundary conditions for different forms of regions under consideration. The complex-variable approach for solving plane problems in homogeneous hollow cylinders was developed by Muskhelishvili [6]. Thermal stresses in cylinders were investigated with the aid of the above-mentioned general solution of the biharmonic equation by Noda, Hetnarski and Tanigawa [7]. The history of the methods developed to solve elasticity and thermoelasticity problems, particularly in polar coordinates, has been discussed in detail in the extended survey by Meleshko [8].

The solution to the plane elasticity problem for a cylinder with radial inhomogeneity in the event when the modulus of elasticity only depends exponentially on the radial coordinate, but the Poisson ratio is assumed constant, was constructed in [9] with the application of the Michell stress function. The authors of that paper also gave recommendations about using their technique for more general cases of material inhomogeneity. The investigation of thermal stresses in a rotating hollow orthotropic cylinder was presented in [10]. To solve the governing equations, the authors applied the proposed numerical method by using a difference scheme. The numerical procedure was used in [11] for finding the non-axisymmetric thermal stresses in a inhomogeneous hollow cylinder. The successive-approximation method has been used in [12] for a three-dimensional stress and displacement analysis of transversely loaded, laminated hollow cylinders and open cylindrical panels. In [13], the analytical solutions to plane elasticity and thermoelasticity problems for a hollow cylinder, whose properties are expressed as a power of the radial coordinate, have been presented in the form of complex Fourier series. The method for solving the inhomogeneous cylinder in the form of a sectioned composite has been treated in [14].

On the basis of a direct integration method, Vihak [15] has developed the technique of solving a plane non-axisymmetric thermoelasticity problem for a solid cylinder. The main feature of this approach is integration of the equilibrium equations. Originally, these equations are in terms of stresses, and they do not depend on physical stress–strain relations, as well as material properties. At the same time, the general equilibrium relates all the stress-tensor components. This makes it possible to express all the stresses in terms of the total stress. Moreover, the integration of the equilibrium equations gives the possibility to deduce the integral equilibrium conditions for all of the stress-tensor components [16]. Such conditions are invariant for different material properties, and they are also necessary for solving the corresponding inverse problems [17] and for proving the accuracy of the calculations. The compatibility equation in terms of strain for the plane non-axisymmetric case with the aid of the physical relations is reduced to a corresponding equation in terms of stresses, which include the total stress only. Such a scheme for the construction of the solution offers ample opportunities for its effective application to inhomogeneous solids. In addition, this approach gives the possibility to avoid the application of biharmonic functions; this is of considerable advantage for inhomogeneous materials, because it does not cause an increase of the order of the governing differential equations.

This paper presents the application of the direct integration method proposed by Vihak for solving plane static non-axisymmetric problems in elasticity and thermoelasticity for a radially inhomogeneous hollow cylinder. Reducing the governing equation to a integral Volterra-type equation of the second kind, we have solved it by Fourier transforms and applying a simple iteration method. The rapid convergence of the iteration process is demonstrated by a numerical example. To determine the stress-tensor components in terms of the obtained total stress, we use the relations between the stresses obtained on the basis of equilibrium equations, which are independent of the material properties.

An analogous technique has been applied to one-dimensional elasticity and thermoelasticity problems for inhomogeneous and thermal-sensitive cylinders [18,19] and to plane problems for a strip that is inhomogeneous with respect to width [20,21].

2 Statement of the problem

Let us consider a hollow circular radially inhomogeneous cylinder subjected to external force loadings on its inner and outer surfaces, as well as to a temperature field distributed within the cylinder. A cylindrical coordinate system r, φ, z is used and the axis of the cylinder coincides with the z-axis. For plane strain, the elastic equilibrium of the cylinder's plane cross-section $\Lambda = \{(\rho, \varphi) \in [k; 1] \times [0; 2\pi]\}$ is governed by [1,5,7]:

(a) equilibrium equations

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho^2\sigma_r\right) + \frac{\partial\sigma_{r\varphi}}{\partial\varphi} = \sigma, \quad \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho^2\sigma_{r\varphi}\right) + \frac{\partial\sigma_\varphi}{\partial\varphi} = 0, \tag{1}$$

(b) the compatibility equation in terms of strains

$$\frac{\partial^2\left(\rho e_{r\varphi}\right)}{\partial\rho\partial\varphi} = \frac{\partial^2 e_r}{\partial\varphi^2} + \frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial e_\varphi}{\partial\rho}\right) - \rho\frac{\partial e_r}{\partial\rho}, \tag{2}$$

(c) physical stress–strain relations

$$e_r = \frac{1-\nu}{2G}\sigma_r - \frac{\nu}{2G}\sigma_\varphi - \nu e_0 + \alpha(1+\nu)T, \quad e_\varphi = \frac{1-\nu}{2G}\sigma_\varphi - \frac{\nu}{2G}\sigma_r - \nu e_0 + \alpha(1+\nu)T, \quad e_{r\varphi} = \frac{1}{G}\sigma_{r\varphi},$$

$$G = \frac{E}{2(1+\nu)}, \tag{3}$$

(d) Cauchy relations

$$e_r = \frac{1}{R_2}\frac{\partial u_r}{\partial\rho}, \quad e_\varphi = \frac{1}{R_2\rho}\left(u_r + \frac{\partial u_\varphi}{\partial\varphi}\right), \quad e_{r\varphi} = \frac{1}{R_2\rho}\left(\frac{\partial u_r}{\partial\varphi} + \rho\frac{\partial u_\varphi}{\partial\rho} - u_\varphi\right). \tag{4}$$

Here $\{\rho, k\} = \{r, R_1\}/R_2$; R_1, R_2 are the internal and external radii of the cylinder; $\sigma_j, \sigma_{r\varphi}$; $e_j, e_{r\varphi}$ $(j = r, \varphi)$ are the components of the stress-tensor and the strain-tensor, respectively; u_r and u_φ denote the radial and angular displacements; $E = E(\rho)$, $G = G(\rho)$ are the elasticity and shear modulus, respectively; $\nu = \nu(\rho)$ is the Poisson ratio, $\alpha = \alpha(\rho)$ is the linear thermal-expansion coefficient; $T(\rho, \varphi)$ is the given temperature field; and the total stress σ is determined by two normal stresses:

$$\sigma = \sigma_r + \sigma_\varphi. \tag{5}$$

It is well known that, if the end sections of long cylinder are free, then, in general, $e_z = e_0 = \text{const} \neq 0$ for the case of plane strain [7]. Thus, the constant axial strain e_0 can be determined from the condition

$$\iint\limits_{\Lambda} \rho\sigma_z \mathrm{d}\rho\mathrm{d}\varphi = 0, \tag{6}$$

and the axial component u_z of the displacement vector is a linear function of the axial coordinate $x = z/R_2$. If the ends of the cylinder are confined between fixed smooth rigid planes, then $e_0 = u_z = 0$. In both cases one has $\sigma_{jz} = e_{jz} = 0 (i = r, \varphi)$; the axial stress can be found from one of the strain–stress relations, namely,

$$\sigma_z = Ee_0 + \nu\sigma - \alpha ET. \tag{7}$$

We solve the system (1)–(4) by taking the external force loadings given at the ring's boundary,

$$\sigma_r|_{\rho=k} = -p_1(\varphi), \quad \sigma_r|_{\rho=1} = -p_2(\varphi), \quad \sigma_{r\varphi}\big|_{\rho=k} = q_1(\varphi), \quad \sigma_{r\varphi}\big|_{\rho=1} = q_2(\varphi), \tag{8}$$

into account.

Using the physical relations (3) and equilibrium equations (1), we present the compatibility equation (2) in terms of the stresses as follows:

$$\Delta\left(\frac{1-\nu}{2G}\sigma + \alpha(1+\nu)T\right) = \frac{\sigma_r}{2}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\left(\frac{1}{G}\right) + \frac{\sigma_\varphi}{2\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G}\right) + \frac{e_0}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}\nu}{\mathrm{d}\rho}\right), \tag{9}$$

where

$$\Delta = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}.$$

3 Construction of the solution

To determine the stress-tensor components from the plane non-axisymmetric elasticity and thermoelasticity problems (1), (8), (9) in terms of the stresses for the ring region Λ, we apply the method proposed in [15]. Following this method, we express the planar stresses in terms of the total stress (5). Applying corresponding differential operations to the equilibrium equations (1), we deduce the three further equations

$$\mathrm{D}\sigma_r = \frac{\partial}{\partial\rho}\left(\rho^2\sigma\right) + \rho\frac{\partial^2\sigma}{\partial\varphi^2}, \quad \mathrm{D}\sigma_\varphi = \rho\frac{\partial^2}{\partial\rho^2}\left(\rho^2\sigma\right), \quad \mathrm{D}\sigma_{r\varphi} = -\rho\frac{\partial^2}{\partial\rho\partial\varphi}(\rho\sigma), \tag{10}$$

where

$$\mathrm{D} = \frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\left(\rho^2\right)\right) + \rho\frac{\partial^2}{\partial\varphi^2}. \tag{11}$$

Let us note that Eqs. (10) are equivalent to Eq. (1) for infinite regions under conditions of stresses that vanish at infinity. But for bounded regions, as the ring region in our case, such an equivalence is possible only under some additional conditions. It is easy to see that the above-mentioned additional conditions should ensure the validity of the equilibrium equations on the region's boundary. For solving the equilibrium and compatibility equations, (1) and (9), respectively, we use the method proposed in [15]. The equations of (10) are used for simplification of the solution construction process which expresses the stresses in terms of the governing function σ.

Following the general scheme proposed in [15], we represent the functions as Fourier series:

$$\begin{Bmatrix} \sigma_r \\ \sigma_\varphi \\ \sigma_z \\ \sigma \\ T \\ p_j \\ \sigma_{r\varphi} \\ q_j \end{Bmatrix} = \begin{Bmatrix} R_0(\rho) \\ \Phi_0(\rho) \\ Z_0(\rho) \\ \sigma_0(\rho) \\ T_0(\rho) \\ p_{j0} \\ S_0(\rho) \\ q_{j0} \end{Bmatrix} + \sum_{n=1}^{\infty}\left[\begin{Bmatrix} R_n^1(\rho) \\ \Phi_n^1(\rho) \\ Z_n^1(\rho) \\ \sigma_n^1(\rho) \\ T_n^1(\rho) \\ p_{jn}^1 \\ S_n^2(\rho) \\ q_{jn}^2 \end{Bmatrix}\cos n\varphi + \begin{Bmatrix} R_n^2(\rho) \\ \Phi_n^2(\rho) \\ Z_n^2(\rho) \\ \sigma_n^2(\rho) \\ T_n^2(\rho) \\ p_{jn}^2 \\ S_n^1(\rho) \\ q_{jn}^1 \end{Bmatrix}\sin n\varphi\right], \quad j = 1, 2, \tag{12}$$

where [22]

$$f_0 = \frac{1}{2\pi}\int_0^{2\pi} f\,\mathrm{d}\varphi, \quad f_\mathrm{c} = \frac{1}{\pi}\int_0^{2\pi} f\ \cos n\varphi\mathrm{d}\varphi, \quad f_\mathrm{s} = \frac{1}{\pi}\int_0^{2\pi} f\ \sin n\varphi\mathrm{d}\varphi, \tag{13}$$

where f is an element of the left-hand column in (12), f_0, f_c, f_s correspond to terms of the series with subscript "0" and coefficients of cos $n\varphi$ and sin $n\varphi$, respectively. Because in (1) the normal stresses are related to the shear stress by the derivative with respect to the angular coordinate φ, we use inverse superscripts for the shear stress and tractions in (12).

By substitution of expressions (12) in (1), (9), (10) and conditions (8) or applying corresponding finite integral transformations prescribed by (13), we deduce the sets of ODEs and boundary conditions to determine the required terms of stress expansions into Fourier series (12). We shall consider below the solution construction for terms of series (12) with subscript "0" (so-called elementary solutions [1,3]) and for those with superscripts "1", "2" separately.

3.1 Construction of elementary solutions

From (1), (5), (10) the equations

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2 R_0\right) = \rho\sigma_0, \quad \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2 S_0\right) = 0, \quad \sigma_0 = R_0 + \Phi_0, \\ &\mathrm{D}_0 R_0 = \frac{\mathrm{d}}{\mathrm{d}\rho}(\rho^2\sigma_0), \quad \mathrm{D}_0\Phi_0 = \rho\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}(\rho^2\sigma_0), \quad \mathrm{D}_0 S_0 = 0, \end{aligned} \tag{14}$$

follow for the elementary parts of the stress expansions (12). Here D_0 is obtained from D operator (11) by exclusion of the derivative with respect to φ. The elementary parts of the radial and shear stresses should satisfy the following boundary conditions:

$$R_0(k) = -p_{10}, \quad R_0(1) = -p_{20}, \quad S_0(k) = q_{10}, \quad S_0(1) = q_{20}, \tag{15}$$

which follow from (8) and (12). In a similar manner, we deduce the key governing equation

$$\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\left(\frac{1-\nu}{2G}\sigma_0 + \alpha(1+\nu)T_0\right)\right) = \frac{\rho R_0}{2}\frac{d^2}{d\rho^2}\left(\frac{1}{G}\right) + \frac{\Phi_0}{2}\frac{d}{d\rho}\left(\frac{1}{G}\right) + e_0\frac{d}{d\rho}\left(\rho\frac{d\nu}{d\rho}\right) \tag{16}$$

on the basis of the compatibility equation (9) in terms of stresses.

It is very simple to find the elementary solution for the shear stresses. From the corresponding equations (14) and last two conditions (15), we conjecture the following solution

$$S_0 = \frac{k^2}{\rho^2}q_{10} = \frac{1}{\rho^2}q_{20} \tag{17}$$

and the conditions

$$q_{20} = k^2 q_{10} \quad \text{or} \quad \int_0^{2\pi} q_2 d\varphi = k^2\int_0^{2\pi} q_1 d\varphi. \tag{18}$$

This solution is discussed in many textbooks on the theory of elasticity [1,4], and others. The solution (17) and conditions (18) correspond to the axisymmetric case, when the shear tractions given on one of the ring's boundaries are balanced by shear tractions prescribed on the other. Also, we conclude that the shear stresses in a radially inhomogeneous cylinder are independent of the material properties for the axisymmetric (one-dimensional) case.

The solution of the system of equations (14) with the first two conditions (15) for the radial stress can be presented in the form

$$\rho^2 R_0 = -k^2 p_{10} + \int_k^{\rho} \eta\sigma_0(\eta)d\eta. \tag{19}$$

By substitution of $\rho = 1$ in the last expression and taking the second condition (15) into account, we obtain the integral condition

$$\int_k^1 \rho\sigma_0(\rho)d\rho = k^2 p_{10} - p_{20} \tag{20}$$

for the elementary part of total stress.

Using integration by parts, we find the solution

$$\sigma_0 = \frac{2G}{1-\nu}\left[C_0 + \nu e_0 - \alpha(1+\nu)T_0 + \frac{1}{2}\int_k^{\rho}\frac{d}{d\eta}\left(\frac{1}{G(\eta)}\right)R_0(\eta)d\eta\right] \tag{21}$$

of Eq. (16), where C_0 is a constant of integration, which can be found with the aid of condition (20), and e_0 can be obtained from (6) and (7) if the ends of the cylinder are free, or $e_0 = 0$ for the case of fixed ends. After substitution of expression (19) in (21), we can find the elementary part of the total stress in the form

$$\sigma_0 = \frac{2G}{1-\nu}\left[C_0 + \nu e_0 - \alpha(1+\nu)T_0 - P_0 + \frac{1}{2}\int_k^{\rho}\frac{1}{\eta^2}\frac{d}{d\eta}\left(\frac{1}{G(\eta)}\right)\int_k^{\eta}\xi\sigma_0(\xi)d\xi d\eta\right], \tag{22}$$

where

$$P_0 = \frac{k^2 p_{10}}{2}\int_k^{\rho}\frac{1}{\eta^2}\frac{d}{d\eta}\left(\frac{1}{G(\eta)}\right)d\eta.$$

By changing the order of integration in integral (22), we reduce this expression to a Volterra-type integral equation

$$\sigma_0 = \frac{2G}{1-\nu}\left[C_0 + \nu e_0 - \alpha(1+\nu)T_0 - P_0 + \frac{1}{2}\int_k^{\rho}\xi\sigma_0(\xi)K_0(\rho,\xi)d\xi\right]. \tag{23}$$

Here

$$K_0(\rho,\xi)=\int_{\xi}^{\rho}\frac{1}{\eta^2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{G(\eta)}\right)\mathrm{d}\eta.$$

We note that (23) agrees with the analogous expression for a one-dimensional problem concerning a radially inhomogeneous cylinder, presented in [18,19]. The authors of the cited works have proposed to solve the integral equation (23) with the aid of a simple iteration method. An analogous procedure is used in [20,21]. According to this, we construct the solution of (23) as the limit

$$\sigma_0=\lim_{m\to\infty}\sigma_0^{(m)}$$

of the approximations

$$\sigma_0^{(m)}=\frac{2G}{1-\nu}\left[C_0^{(m)}+\nu e_0^{(m)}-\alpha(1+\nu)T_0-P_0+\frac{1}{2}\int_k^{\rho}\xi\sigma_0^{(m-1)}(\xi)K_0(\rho,\xi)\mathrm{d}\xi\right],\tag{24}$$

where $m=1,2,\ldots,\sigma_0^{(0)}\equiv 0$. In the case of free ends of the cylinder, the constants $C_0^{(m)}$, $e_0^{(m)}$ can be found from conditions (6), (7), (20) in the form

$$C_0^{(m)}=\frac{1}{\gamma_0}\left[I_1b_1^{(m)}-I_2b_2^{(m)}\right],\quad e_0^{(m)}=\frac{1}{\gamma_0}\left[I_1b_2^{(m)}-I_2b_1^{(m)}\right],$$

where

$$b_1^{(m)}=k^2p_{10}-p_{20}+2\int_k^1\frac{\rho G}{1-\nu}\left(\alpha(1+\nu)T_0+P_0\right)\mathrm{d}\rho-\int_k^1\frac{\rho G}{1-\nu}\int_k^{\rho}\xi\sigma_0^{(m-1)}(\xi)K_0(\rho,\xi)\mathrm{d}\xi\,\mathrm{d}\rho,$$

$$b_2^{(m)}=\int_k^1\rho\alpha ET_0\,\mathrm{d}\rho+2\int_k^1\frac{\rho\nu G}{1-\nu}\left(\alpha(1+\nu)T_0+P_0\right)\mathrm{d}\rho-\int_k^1\frac{\rho\nu G}{1-\nu}\int_k^{\rho}\xi\sigma_0^{(m-1)}(\xi)K_0(\rho,\xi)\mathrm{d}\xi\mathrm{d}\rho,$$

$$\gamma_0=2\left(I_1^2-I_2^2\right)=G^*E_1,\quad E_1=\int_k^1\frac{\rho E}{1-\nu}\mathrm{d}\rho,\quad G^*=\int_k^1\rho G\mathrm{d}\rho,\quad I_1=\int_k^1\frac{\rho G}{1-\nu}\mathrm{d}\rho,\quad I_2=\int_k^1\frac{\rho\nu G}{1-\nu}\mathrm{d}\rho.$$

If the ends are clamped, then $e_0^{(m)}=0$ in (24), and the expression

$$C_0^{(m)}=\frac{b_1^{(m)}}{2I_1}$$

follows from (20).

After having found the elementary part of the total stress, we determine R_0 by (19). The elementary constituent of the angular stress Φ_0 can be calculated by means of the third equation of (14)

$$\Phi_0=\sigma_0-R_0.\tag{25}$$

By taking expression (19) into account, it is easy to show that (25) is also the solution to the fifth equation of (14).

3.2 Angle-dependent parts of the solution

Now we apply the last two integral transforms (13) to Eqs. (1), (5), (9) and (10), and the boundary conditions (8), thus obtaining the equations

$$\begin{gathered}\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2R_n^i\right)-(-1)^inS_n^i=\sigma_n^i,\quad \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2S_n^i\right)+(-1)^in\Phi_n^i=0,\quad \sigma_n^i=R_n^i+\Phi_n^i,\\ \mathrm{D}_nR_n^i=\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\sigma_n^i\right)-n^2\rho\sigma_n^i,\quad \mathrm{D}_n\Phi_n^i=\rho\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\left(\rho^2\sigma_n^i\right),\quad \mathrm{D}_nS_n^i=-(-1)^in\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\sigma_n^i\right),\end{gathered}\tag{26}$$

$$\Delta_n\left[\frac{1-\nu}{2G}\sigma_n^i+\alpha(1+\nu)T_n^i\right]=\frac{R_n^i}{2}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\left(\frac{1}{G}\right)+\frac{\Phi_n^i}{2\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G}\right)\tag{27}$$

and boundary conditions

$$R_n^i(k) = -p_{1n}^i, \quad R_n^i(1) = -p_{2n}^i, \quad S_n^i(k) = q_{1n}^i, \quad S_n^i(1) = q_{2n}^i, \tag{28}$$

where $\mathrm{D}_n = \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\right)\right) - n^2\rho$, $\Delta_n = \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\right) - \frac{n^2}{\rho^2}$, $i = 1, 2$.

Following a technique proposed by Vihak [15], from the set of equations (26) and conditions (28) we get expressions for the stress-tensor components:

$$S_n^i = \frac{1}{2\rho^2}\left[k^2\chi_n^+(\rho,k)q_{1n}^i + (-1)^i k^2\chi_n^-(\rho,k)p_{1n}^i - (-1)^i\int_k^\rho \eta\sigma_n^i(\eta)\left(n\chi_n^+(\rho,\eta) + \chi_n^-(\rho,\eta)\right)\mathrm{d}\eta\right],$$

$$R_n^i = \frac{-1}{2\rho^2}\left[(-1)^i k^2\chi_n^-(\rho,k)q_{1n}^i + k^2\chi_n^+(\rho,k)p_{1n}^i - \int_k^\rho \eta\sigma_n^i(\eta)\left(n\chi_n^-(\rho,\eta) + \chi_n^+(\rho,\eta)\right)\mathrm{d}\eta\right], \tag{29}$$

$$\Phi_n^i = \frac{1}{2\rho^2}\left[(-1)^i k^2\chi_n^-(\rho,k)q_{1n}^i + k^2\chi_n^+(\rho,k)p_{1n}^i - \int_k^\rho \eta\sigma_n^i(\eta)\left(n\chi_n^-(\rho,\eta) + \chi_n^+(\rho,\eta)\right)\mathrm{d}\eta\right] + \sigma_n^i$$

in terms of the total stress, as well as the integral conditions

$$\begin{aligned}(n+1)\int_k^1 \rho^{n+1}\sigma_n^i\mathrm{d}\rho &= k^{n+2}p_{1n}^i - p_{2n}^i + (-1)^i\left(k^{n+2}q_{1n}^i - q_{2n}^i\right),\\ (n-1)\int_k^1 \rho^{-n+1}\sigma_n^i\mathrm{d}\rho &= p_{2n}^i - k^{-n+2}p_{1n}^i + (-1)^i\left(k^{-n+2}q_{1n}^i - q_{2n}^i\right).\end{aligned} \tag{30}$$

Here $i = 1, 2$; $\chi_n^+(a,b) = a^{-n}b^n + a^n b^{-n}$, $\chi_n^-(a,b) = a^{-n}b^n - a^n b^{-n}$. We shall use (30) to determine the unknowns in the expressions for the total-stress coefficients $\sigma_n^i (i = 1, 2)$, which will be found from (27).

Note that the second condition (30) is degenerate for $n = 1$, making it impossible to use this condition for the elimination of the multivalued constituents in the expression for the total-stress coefficients. For this case, to complete the set of conditions for the determination of the constants of integration, we shall deduce one further condition below. However, by assuming $n = 1$ in the second condition (30), we get the necessary conditions

$$kp_{11}^i - p_{21}^i = (-1)^i\left(kq_{11}^i - q_{21}^i\right), \quad i = 1, 2,$$

which, with the aid of (13), take the form

$$k\int_0^{2\pi}(p_1 + q_1)\cos\varphi\,\mathrm{d}\varphi = \int_0^{2\pi}(p_2 + q_2)\cos\varphi\,\mathrm{d}\varphi, \quad k\int_0^{2\pi}(p_1 - q_1)\sin\varphi\,\mathrm{d}\varphi = \int_0^{2\pi}(p_2 - q_2)\sin\varphi\,\mathrm{d}\varphi \tag{31}$$

in terms of the external force loadings. As is easy to see, the conditions (31) express the equilibrium of the external forces acting on the ring in projections onto the axes of Cartesian coordinates [1, p. 134].

The solution of Eq. (27) can be given as follows:

$$\begin{aligned}\frac{1-\nu}{2G}\sigma_n^i + \alpha(1+\nu)T_n^i = {} & A_n^i\rho^{-n} + B_n^i\rho^n - \frac{k\chi_n^-(\rho,k)p_{1n}^i}{4n}\left.\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G}\right)\right|_{\rho=k}\\ & + \frac{1}{4}\int_k^\rho \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{G(\eta)}\right)R_n^i(\eta)\chi_n^+(\rho,\eta)\mathrm{d}\eta\\ & + \frac{1}{4n}\int_k^\rho \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{G(\eta)}\right)\frac{\chi_n^-(\rho,\eta)}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^2 R_n^i(\eta)\right)\mathrm{d}\eta\\ & - \frac{1}{4n}\int_k^\rho \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{G(\eta)}\right)\chi_n^-(\rho,\eta)\sigma_n^i(\eta)\mathrm{d}\eta,\end{aligned} \tag{32}$$

where $A_n^i, B_n^i (i = 1, 2; n = 1, 2, \ldots)$ denote constants of integration.

Substituting expression (29) for R_n^i in (32) and using integration by parts, we find

$$\frac{1-\nu}{2G}\sigma_n^i + \alpha(1+\nu)T_n^i = A_n^i\rho^{-n} + B_n^i\rho^n + P_n^i + Q_n^i + \frac{1}{4}\int_k^\rho \frac{1}{\eta^2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{G(\eta)}\right) \times \int_k^\eta \xi\sigma_n^i(\xi)\left((n+1)\rho^n\xi^n\eta^{-2n} - (n-1)\rho^{-n}\xi^{-n}\eta^{2n}\right)\mathrm{d}\eta. \quad (33)$$

Here $i = 1, 2$; $n = 1, 2, \ldots$;

$$P_n^i = -\frac{kp_{1n}^i}{4}\left(\frac{\chi_n^-(\rho,k)}{n}\,\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G}\right)\bigg|_{\rho=k} + k\int_k^\rho \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{G(\eta)}\right)\left(k^{-n}\rho^{-n}\eta^{2(n-1)} + k^n\rho^n\eta^{-2(n+1)}\right)\mathrm{d}\eta\right),$$

$$Q_n^i = \frac{(-1)^i k^2 q_{1n}^i}{4}\int_k^\rho \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{G(\eta)}\right)\left(k^{-n}\rho^{-n}\eta^{2(n-1)} - k^n\rho^n\eta^{-2(n+1)}\right)\mathrm{d}\eta.$$

The Volterra-type integral equation,

$$\sigma_n^i = \frac{2G}{1-\nu}\left[A_n^i\rho^{-n} + B_n^i\rho^n + P_n^i + Q_n^i - \alpha(1+\nu)T_n^i + \frac{1}{4}\int_k^\rho \xi\sigma_n^i(\xi)K_n(\rho,\xi)\mathrm{d}\xi\right], \quad (34)$$

follows from (33) by means of changing the order of intergation. Here the kernel of the integral is given by

$$K_n(\rho,\xi) = \int_\xi^\rho \frac{1}{\eta^2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{G(\eta)}\right)\left((n+1)\rho^n\xi^n\eta^{-2n} - (n-1)\rho^{-n}\xi^{-n}\eta^{2n}\right)\mathrm{d}\eta.$$

To solve the integral equation (34), we use the scheme given above for the elementary solutions. Thus, the total-stress coefficients can be found as the limit

$$\sigma_n^i = \lim_{m\to\infty}\sigma_n^{i(m)}$$

of the iterations

$$\sigma_n^{i(m)} = \frac{2G}{1-\nu}\left[A_n^{i(m)}\rho^{-n} + B_n^{i(m)}\rho^n + P_n^i + Q_n^i - \alpha(1+\nu)T_n^i + \frac{1}{4}\int_k^\rho \xi\sigma_n^{i(m-1)}(\xi)K_n(\rho,\xi)\mathrm{d}\xi\right], \quad (35)$$

where $m = 1, 2, \ldots$; $\sigma_n^{i(0)} \equiv 0$; $i = 1, 2$; $n = 1, 2, \ldots$. For $n > 1$ the constants $A_n^{i(m)}$, $B_n^{i(m)}$ take the form

$$A_n^{i(m)} = \frac{1}{\gamma_n}\left[I_1F_{1n}^{i(m)} - I_n^+F_{2n}^{i(m)}\right], \quad B_n^{i(m)} = \frac{1}{\gamma_n}\left[I_1F_{2n}^{i(m)} - I_n^-F_{1n}^{i(m)}\right],$$

which follow from conditions (30). Here

$$F_{1n}^{i(m)} = \frac{1}{2(n+1)}\left[k^{n+2}p_{1n}^i - p_{2n}^i + (-1)^i\left(k^{n+2}q_{1n}^i - q_{2n}^i\right)\right] - \int_k^1 \frac{\rho^{n+1}G}{1-\nu}\left[P_n^i + Q_n^i - \alpha(1+\nu)T_n^i + \frac{1}{4}\int_k^\rho \xi\sigma_n^{i(m-1)}(\xi)K_n(\rho,\xi)\mathrm{d}\xi\right]\mathrm{d}\rho,$$

$$F_{2n}^{i(m)} = \frac{1}{2(n-1)}\left[-k^{-n+2}p_{1n}^i + p_{2n}^i + (-1)^i\left(k^{-n+2}q_{1n}^i - q_{2n}^i\right)\right] - \int_k^1 \frac{\rho^{-n+1}G}{1-\nu}\left[P_n^i + Q_n^i - \alpha(1+\nu)T_n^i + \frac{1}{4}\int_k^\rho \xi\sigma_n^{i(m-1)}(\xi)K_n(\rho,\xi)\mathrm{d}\xi\right]\mathrm{d}\rho,$$

$$\gamma_n = I_1^2 - I_n^-I_n^+, \quad I_1 = \int_k^1 \frac{\rho G}{1-\nu}\mathrm{d}\rho, \quad I_n^- = \int_k^1 \frac{\rho^{1-2n}G}{1-\nu}\mathrm{d}\rho, \quad I_n^+ = \int_k^1 \frac{\rho^{1+2n}G}{1-\nu}\mathrm{d}\rho.$$

As additional conditions for determining the constants $A_1^{i(m)}$, $B_1^{i(m)}$, we shall use the so-called integral compatibility conditions

$$\int_0^{2\pi}\left[ke_r(k) + e_r(1) - k^2\frac{\partial e_\varphi}{\partial\rho}\bigg|_{\rho=k} - \frac{\partial e_\varphi}{\partial\rho}\bigg|_{\rho=1}\right]\begin{Bmatrix}\sin\varphi\\ \cos\varphi\end{Bmatrix}\mathrm{d}\varphi = \int_0^{2\pi}\left[ke_{r\varphi}(k) + e_{r\varphi}(1)\right]\begin{Bmatrix}\cos\varphi\\ -\sin\varphi\end{Bmatrix}\mathrm{d}\varphi, \quad (36)$$

deduced in [23] on the basis of the integration of the geometric Cauchy relations (4). Here we denote $e_r(k) = e_r(k,\varphi)$, etc. for the sake of brevity. Using the relations (3), the expressions for the stresses (29), (34), and the formulas (13), on the basis of the conditions (36), we derive the expressions

$$\begin{aligned}4A_1^{i(m)} &= \frac{(-1)^i q_{21}^i}{2G(1)} + \frac{(-1)^i k q_{11}^i}{2}\left[\frac{2}{G(k)} - \frac{1}{G(1)} + \frac{1-k^2}{2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G(\rho)}\right)\Big|_{\rho=1}\right] \\ &+\frac{kp_{11}^i}{2}\left[\frac{1}{G(1)} - \frac{2}{G(k)} + 2k\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G(\rho)}\right)\Big|_{\rho=k} - \frac{1+k^2}{2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G(\rho)}\right)\Big|_{\rho=1}\right] \\ &-\frac{p_{21}^i}{2}\left[\frac{1}{G(\rho)} - \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G(\rho)}\right)\right]_{\rho=1} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{1}{G(\rho)}\right)\Big|_{\rho=1}\int_k^1 \rho^2\sigma_1^{i(m-1)}\mathrm{d}\rho, \quad i=1,2. \end{aligned} \tag{37}$$

The expressions for the constants $B_1^{i(m)}$

$$\begin{aligned}4I_3 B_1^{i(m)} &= -4I_1A_1^{i(m)} + k^3p_{11}^i - p_{21}^i + (-1)^i\left(k^3q_{11}^i - q_{21}^i\right) \\ &-4\int_k^1 \frac{\rho^2 G}{1-\nu}\left\{P_1^i + Q_1^i - \alpha(1+\nu)T_1^i + \frac{1}{4}\int_k^\rho \xi\sigma_1^{i(m-1)}(\xi)K_1(\rho,\xi)\mathrm{d}\xi\right\}\mathrm{d}\rho\end{aligned}$$

follow from the first condition of (30). Here $i = 1, 2$; $m = 1, 2, \ldots$;

$$I_3 = \int_k^1 \frac{\rho^3 G}{1-\nu}\mathrm{d}\rho;$$

and $A_1^{i(m)}$ is presented by (37).

With the determination of the constants $A_1^{i(m)}$, $B_1^{i(m)}$, we have completed the construction of the solution to the stated plane problems of elasticity and thermoelasticity for a hollow circular cylinder consisting of radially inhomogeneous material.

After having found the planar stress-tensor components, we find the terms Z_0, $Z_n^i (i = 1, 2)$ of the expansions (12) of the axial stress into Fourier series from formula (7). We point out that the axial stress satisfies condition (6). But it is easy to see that, in general, the cylinder is under the action of bending moments applied on its ends. To release the cylinder from such reactive moments, it is necessary to apply moments of the same magnitude but of opposite direction on its ends. Such moments can be found easily with the aid of the technique developed in [15]. It is necessary to take this into account for a correct determination of the axial stress.

It is easy to conclude from (24), (35) that these formulas give the exact solutions already for $m = 1$, if G is a constant.

3.3 Static temperature field

The coefficients (12) of the static temperature satisfying the equation

$$\Delta T = 0,$$

can be found in the form

$$T_0 = a_0 + b_0 \ln\,\rho, \quad T_n^i = a_n^i\rho^n + b_n^i\rho^{-n}, \quad i = 1, 2; \quad n = 1, 2, \ldots . \tag{38}$$

The constants a_0, b_0, a_n^i, b_n^i have to be determined from boundary conditions prescribed for the temperature field.

4 Numerical examples and discussion

4.1 Example 1

Let us consider external tractions (8) of the form

$$p_1 = q_1 = q_2 = 0, \quad p_2 = \begin{cases} p, \varphi \in V, \\ 0, \varphi \notin V, \end{cases} \tag{39}$$

where $V = \left[\frac{(2l+1)\pi}{2} - \Theta, \frac{(2l+1)\pi}{2} + \Theta\right]$; $l = 0, 1$; $p = \text{const}$; $\Theta \in \left(0, \frac{\pi}{2}\right]$. The temperature field is assumed to be $T = 0$. This example of uniform pressure given on two equal but opposite parts of the outer boundary of the ring Λ has been widely discussed [2,4] for homogeneous materials. The case of a concentrated force acting on polar points of the outer boundary of the homogeneous ring (we can approximate this type of force loadings by assuming Θ to be small in our example) has been studied theoretically [1–5], as well as experimentally, by methods of photoelasticity [24].

We give a calculation of the stresses assuming that the elasticity modulus varies exponentially, as $E(\rho) = E_0 e^{\beta\rho}$ ($E_0, \beta = \text{const}$), and $\nu = \nu_0 = \text{const}$. According to the expression for the shear modulus (3), the latter takes the same form as the elasticity modulus, namely, $G(\rho) = G_0 e^{\beta\rho}$, where $G_0 = E_0/(2(1+\nu_0)) = \text{const}$. It should be noted that the case $\beta = 0$ corresponds to homogeneous properties of the material. We have studied the stress distributions in ring Λ for $\beta = -2; 0; 2$, using the Poisson ratio $\nu_0 = 0.3$; the inner radius of the cylinder is $k = 0.5$; the angle is $\Theta = \frac{\pi}{4}$. We also assume that the cylinder's ends are rigidly clamped; thus, $e_0 = 0$.

Figure 1 shows how the computed dimensionless total stress σ/p depends on the radial coordinate with respect to cylinder thickness in the angular cross-sections $\varphi = 0$ and $\varphi = \frac{\pi}{2}$. The specified angular cross-sections $\varphi = \frac{\pi}{2}$ and $\varphi = 0$ are the symmetry axis of the ring's sectors corresponding to the loaded and free parts, respectively, of the outer boundary $\rho = 1$. We observe that the coordinate dependence of the Young's modulus of the material has an enormous effect on the distribution of the total stress.

The calculations show a rapid convergence of the iterative scheme described above. Table 1 presents the maximum relative error, $\varepsilon_m = \max_{\rho\in[k,1]} \frac{|\sigma_0^{(m)} - \sigma_0^{(m-1)}|}{|\sigma_0^{(m-1)}|} \times 100(\%)$, of the iterations (24) for $\sigma_0^{(m)}$ in the case of loading (39). From this we conclude that, to obtain a high degree of accuracy, it is enough to complete three or four iterations (24) for the elementary part of the total stress. An analogous investigation for iterations (35) shows that, to obtain an accuracy to within 0.001 in iterations $\sigma_n^{(m)}$ for $1 \le n \le 15$, it is sufficient to take three ($m = 3$) iterations (35);

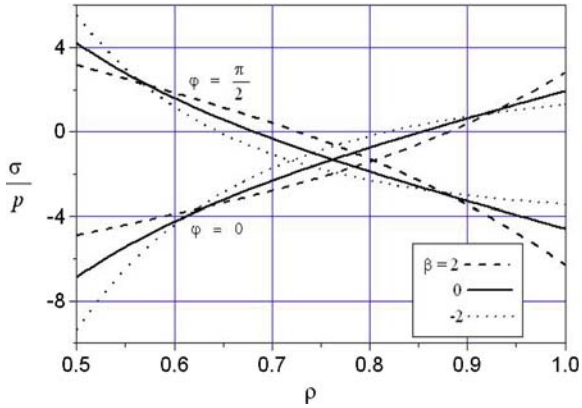

Fig. 1 ρ-Dependence of the dimensionless total stress in the some angular cross-sections for different cases of material inhomogeneity

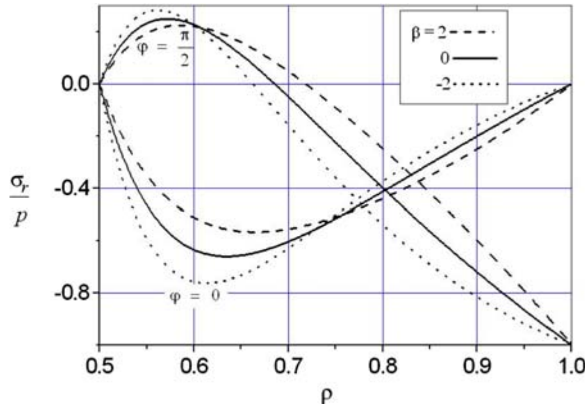

Fig. 2 The influence of material inhomogeneity on the radial stress

Table 1 The maximum relative error of the iterations $\sigma_0^{(m)}$

m	ε_m (%) $\beta = 2$	$\beta = -2$
2	15.31	24.53
3	1.40	2.11
4	0.13	0.21
5	0.01	0.02

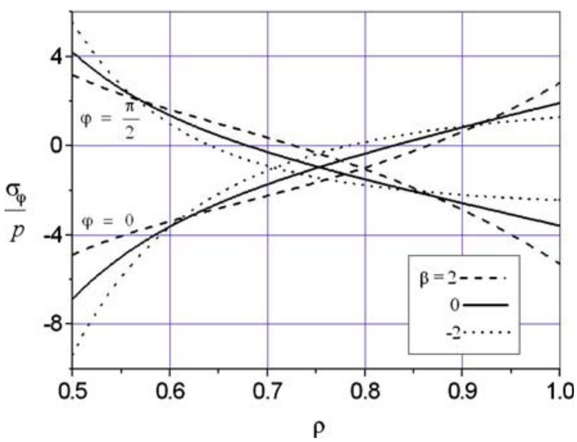

Fig. 3 ρ-Dependence of the angular stress

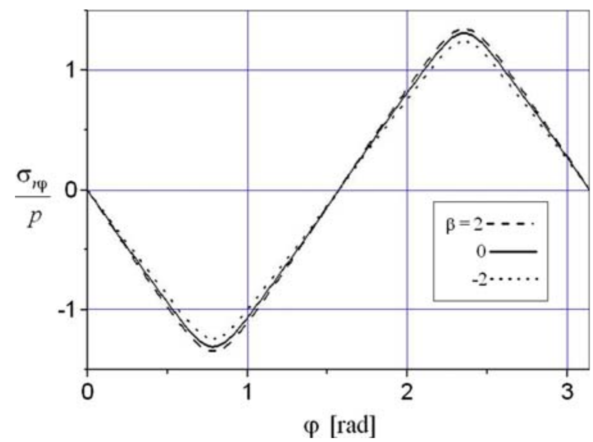

Fig. 4 Distribution of the shear stresses on the mid-surface of the cylinder

the second iteration gives the mentioned accuracy for $15 < n$. The results presented below have been computed by taking 140 terms in the series (12) into account.

Figure 2 depicts the distribution of the radial stress along the thickness of the cylinder in the same cross-sections as in the previous figure. We observe that this stress, calculated by the corresponding formulas (19), (29) under a known total stress, satisfies the boundary conditions exactly. The calculations show that the intensity of this stress increases with decreasing cylinder thickness. The radial stress takes positive values near the inner boundary in the sectors $\varphi \in V$ corresponding to the loaded parts of the outer boundary. These positive values of the stresses, as well as their distribution zones, increase with decreasing cylinder thickness. In the sectors corresponding to the unloaded parts of outer boundary, the radial stress takes maximal negative values near the inner boundary too. Finally, we emphasize the essential influence of material inhomogeneity on the distribution of the radial stresses.

The influence of the radial variation of Young's modulus on the distribution of the dimensionless angular stress is shown in Fig. 3. As can be seen, the maximum (positive in loaded sectors and negative elsewhere) values of this stress are located on the inner boundary. This stress intensity increases with decreasing cylinder thickness. Moreover, the absolute values of the angular stresses exceed the corresponding values of the radial stress. It is known that the peaks of the shear stresses are located in the neighbourhood of singular points of the normal tractions [6]. Figure 4 demonstrates the dimensionless shear stress $\sigma_{r\varphi}/p$ on the mid-surface, $\rho = (1 + k)/2 = 0.75$, of the cylinder.

4.2 Example 2

Let us consider now an example of computing the thermal stresses caused by a temperature field T only (in the absence of external force loadings on the inner and outer surfaces of the cylinder). We find the constants in the

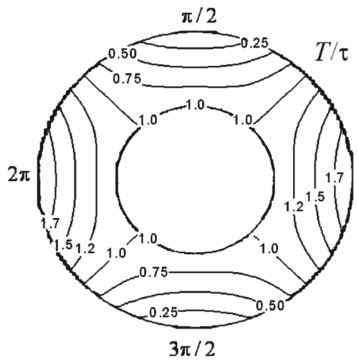

Fig. 5 Distribution of the dimensionless temperature (41) in the ring Λ

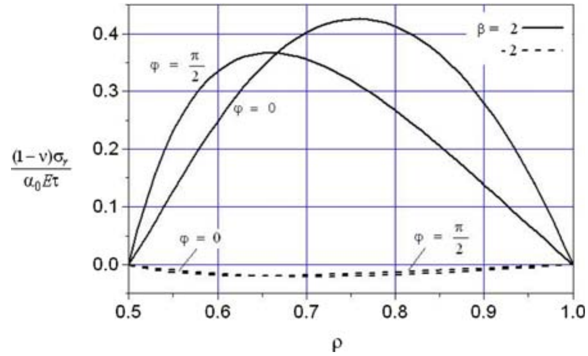

Fig. 6 Distribution of the dimensionless radial thermal stresses along the radius of the cylinder in the cross-sections $\varphi = 0;\ \frac{\pi}{2}\ (k = 0.5)$

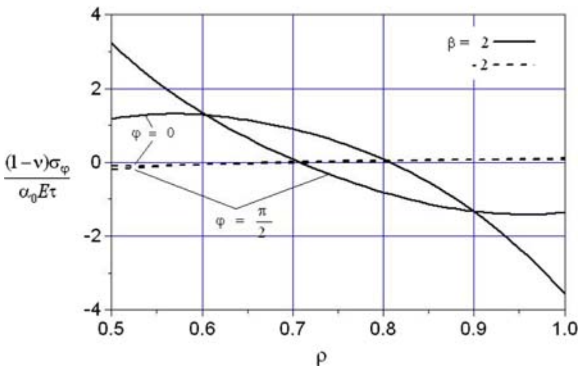

Fig. 7 The dimensionless angular thermal stresses in the cross-sections $\varphi = 0;\ \frac{\pi}{2}\ (k = 0.5)$

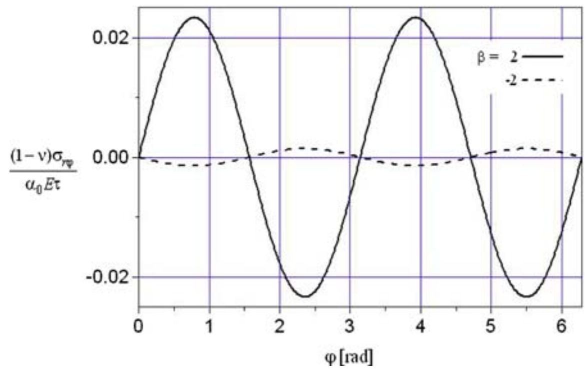

Fig. 8 Distribution of the shear thermal stresses on the mid-surface $\rho = 0.75$ of the cylinder ($k = 0.5$)

expressions (38) from the boundary conditions:

$$T(k,\varphi) = \tau = \text{const}, \quad T(1,\varphi) = \tau\,(1 + \cos 2\varphi)\,. \tag{40}$$

The temperature (12) with the coefficients (38) subjected to the boundary conditions (40) takes the form

$$\frac{T}{\tau} = 1 + \frac{\rho^2 - k^4\rho^{-2}}{1 - k^4}\cos 2\varphi. \tag{41}$$

Figure 5 depicts the distribution of the dimensionless temperature (41) for an inner radius $k = 0.5$. The calculations of the thermal stresses caused by the temperature (41) are obtained under the assumption that $\alpha = \alpha_0 e^{\beta\rho}, \quad \{G, \nu, E, \alpha_0, \beta\} = \text{const}$. As was mentioned above, the formulas (24) and (35) express the exact solutions already for $m = 1$ in this case.

It is easy to prove [25] that there are no stresses distributed in region Λ if $\beta = 0$ (homogeneous material) for the considered example. But for a variable coefficient of linear thermal expansion ($\beta \neq 0$) the temperature field (41) causes stresses in Λ. Figures 6–8 show the computed dimensionless thermal stresses $(1-\nu)\{\sigma_r, \sigma_\varphi, \sigma_{r\varphi}\}/(\alpha_0 E\tau)$, respectively, for the parameter values $\beta = 2$ and $\beta = -2$. We observe the essential influence of the radial dependence of the coefficient α on the distribution of the thermal stresses.

5 Conclusions

To develop a technique for the solution of static plane non-axisymmetric elasticity and thermoelasticity problems for a radially inhomogeneous cylinder, we have used Vihak's method for solving the analogous problem for a homogeneous solid cylinder. Due to the derived relations between the stress-tensor components, we can simplify the calculation of the stress state in inhomogeneous cylinders considerably, because these relations do not depend on the material properties.

The rapid convergence of the iterations can be explained by the fact that, in the particular case of a homogeneous material, the initial approximation gives the exact analytical solution of the corresponding elasticity or thermoelasticity problem. Because the presented technique shows a rapid convergence of the iterative process, namely, the second or third iteration can be accepted with 'engineering' accuracy, it is easy to complete the approximate analytical expression for the total stress by taking two or three iterations only. This expression promises to be quite useful for practical calculations and analytical investigations.

Acknowledgements The authors gratefully acknowledge the financial support of this research by the National Science Council (Republic of China) under Grant NSC 95-2221-E-002-048-MY2.

References

1. Timoshenko SP, Goodier JN (1970) Theory of elasticity. McGraw-Hill Book Company, New York
2. Little RWm (1973) Elasticity. Prentice-Hall, Inc., Englewood Cliffs, NJ
3. Lurie AI (1964) Three-dimensional problems of the theory of elasticity. Interscience Pub., New York
4. Papkovich PF (1939) Theory of elasticity in Russian. Oborongiz, Leningrad-Moscow
5. Sadd MH (2005) Elasticity: theory, applications, and numerics. Elsevier, Amsterdam
6. Muskhelishvili NI (1963) Some basic problems of the mathematical theory of elasticity. P. Noordhoff Ltd, Groningen
7. Noda N, Hetnarski RB, Tanigawa Yo (2000) Thermal stresses. Lastran Corp, Rochester, NY
8. Meleshko VV (2003) Selected topics in the history of the two-dimensional biharmonic problem. Appl Mech Rev 56(1): 33–85
9. Zhang X, Hasebe N (1999) Elasticity solution for a radially nonhomogeneous hollow circular cylinder. J Appl Mech Trans ASME 66:598–606
10. El-Naggar AM, Abd-Alla AM, Fahmy MA, Ahmed SM (2002) Thermal stresses in a rotating non-homogeneous orthotropic hollow cylinder. Heat Mass Transfer 39:41–46
11. Vasilenko AT (1995) Nonaxisymmetric thermal stressed state of nonhomogeneous anisotropic cylinders. Int Appl Mech 31(11):895–899
12. Ye J, Soldatos KP (1994) Three-dimensional stress analysis of orthotropic and cross-ply laminated hollow cylinders and cylindrical panels. Comput Methods Appl Mech Eng 117:331–351
13. Jabbari M, Sohrabpour S, Eslami MR (2003) General solution for mechanical and thermal stresses in a functionally graded hollow cylinder due to nonaxisymmetric steady-state loads. J Appl Mech Trans ASME 70:111–118
14. Liew KM, Kitipornchai S, Zhang XZ, Lim CW (2003) Analysis of the thermal stress behaviour of functionally graded hollow circular cylinders. Int J Solid Struc 40:2355–2380
15. Vihak VM (1996) Solution of the thermoelastic problem for a cylinder in the case of a two-dimensional nonaxisymmetric temperature field. Z Angew Math Mech 76(1):35–43
16. Vihak VM, Pushak YaS (1999) Integral equilibrium conditions for stresses of two-dimensional non-axisymmetric problems of mechanics of solids in ring domains in Ukrainian. Visnyk NU Lvivska Politekhnika, ser Prykladna mathematyka 364:74–83
17. Vihak VM (1988) Control of thermal stresses and displacements in Russian. Naukova Dumka, Kiev, pp. 312
18. Vihak VM, Kalyniak BM (1999) Reduction of one-dimensional elasticity and thermoelasticity problems in inhomogeneous and thermal sensitive solids to the solution of integral equation of Volterra type. Proceedings of the 3rd international congress "thermal stresses-99", Cracow, Poland, pp 457–460
19. Kalynyak BM (2000) Integration of equations of one-dimensional problems of elasticity and thermoelasticity for inhomogeneous cylindrical bodies. J Math Sci 99(5):1662–1670
20. Tokovyy YuV, Rychahivskyy AV (2005) Reduction of plane thermoelasticity problem in inhomogeneous strip to integral Volterra type equation. Math Modell Anal 10(1):91–100
21. Tokovyi YuV, Rychahivs'kyi AV (2005) Analytic solution of the plane problem of the theory of elasticity for nonuniform strip. Mater Sci 41(1):135–138

22. Tolstov GP (1962) Fourier series. Prentice-Hall, Englewood Cliffs, NJ
23. Vihak VM, Pushak YaS (2000) Integration of compatibility equations for two-dimensional non-axisymmetric problems of mechanics in circular domains in Ukrainian. Visnyk NU Lvivska Politekhnika, ser Prykladna mathematyka 407:42–51
24. Frocht MM (1948) Photoelasticity, vol 1. Willey, New York
25. Kovalenko AD (1969) Thermoelasticity: basic theory and applications. Wolters-noordhoff Publishing, Groningen